A generalized Cauchy process and its application to relaxation phenomena

2006 J. Phys. A: Math. Gen. 39 2935

(http://iopscience.iop.org/0305-4470/39/12/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 2935–2951 doi:10.1088/0305-4470/39/12/005

A generalized Cauchy process and its application to relaxation phenomena

S C Lim[1] **and Ming Li**[2]

[1] Faculty of Engineering, Multimedia University, 63100 Cyberjaya, Selangor, Malaysia
[2] School of Information Science & Technology, East China Normal University, Shanghai 200026, People's Republic of China

E-mail: sclim@mmu.edu.my, mli@ee.ecnu.edu.cn and ming_lihk@yahoo.com

Received 16 August 2005, in final form 7 February 2006
Published 8 March 2006
Online at stacks.iop.org/JPhysA/39/2935

Abstract

We study some of the basic properties of a generalized Cauchy process indexed by two parameters. The application of the Lamperti transformation to the generalized Cauchy process leads to a self-similar process which preserves the long-range dependence. The asymptotic properties of spectral density of the process are derived. Possible application of this process to model relaxation phenomena is considered.

PACS numbers: 02.50.Ey, 05.45.Tp, 77.22.Gm

1. Introduction

Recently, Gneiting and Schlather [1] introduced a class of stationary Gaussian processes indexed by two parameters. They called such processes the Cauchy class since these processes can be regarded as an extension of the generalized Cauchy process used in geostatistics [2]. For simplicity, we call a process belonging to such a class a generalized Cauchy (GC) process. The covariance of the generalized Cauchy process has the same analytic form as the characteristic function of the generalized Linnik distribution [3–5], just like in the case of the stretched exponential process which has covariance in the same functional form as the characteristic function of the symmetric Levy stable distribution [6]. The recent increase in interest in the generalized Linnik distribution is attributed mainly to the heavy-tailed properties of Linnik laws [7], which have potential applications in many areas ranging from anomalous diffusion [8] to financial time series [9]. One of the reasons that the generalized Linnik distribution is not as widely used as the stable distribution is because of the general acceptance of the latter as the model for heavy-tailed phenomena [10]. It is also interesting to note that just like in the case of stable distribution where there exist many physical systems that obey laws in the same functional form as the stretched exponential law (characteristic function of symmetric stable distribution), there are also laws in physics which have the same analytic form as the characteristic function of the generalized Linnik distribution or the covariance of the GC process. The most notable one is the Havriliak–Negami relaxation law in the non-Debye relaxation theory [11, 12]. Thus, results obtained in any one of these three areas, namely the generalized Linnik distribution, the Havriliak–Negami relaxation law and the GC process, are of relevance to the other two.

The main aim of this paper is to study the properties of the GC process in more detail in view of its potential applications in modelling long-range-dependent (LRD) phenomena which exist in many physical, biological, teletraffic and economical systems. Currently, one of the most widely used models for LRD is based on fractional Gaussian noise (fGn), which can be regarded as the (generalized) derivative process of fractional Brownian motion (fBm) [13]. Both fGn and fBm are characterized by a single parameter called the Hurst index H, with $0 < H < 1$. In actual application, it is rather difficult to characterize the covariance function of a Gaussian random process over the entire trace by a single parameter [14]. In the case of fGn, it has the weakness of not being able to describe accurately the covariance of the actual process for the short time lag. Furthermore, in the fGn (or fBm) model, the fractal dimension (or self-similarity) which is local in nature and the global LRD property are both determined by a single Hurst index. Therefore, it may be useful to have a model based on a random process which allows separate characterization of the LRD and self-similar properties. Such a process indexed by two parameters may offer a more flexible model.

In section 2, we study some basic properties of the GC process. The self-similar process associated with the GC process obtained by the Lamperti transformation is considered in section 3. In the subsequent section, we study the asymptotic properties of the spectral density of the GC process. The possible application of the GC process to model relaxation phenomena is discussed in section 5, which is followed by the conclusion.

2. The generalized Cauchy process indexed by two parameters

$X(t)$ is called a GC process if it is a stationary Gaussian-centred process with the following covariance [1]:

$$C(t) = \langle X(s+t)X(s)\rangle = (1 + |t|^{\alpha})^{-\beta}, \qquad t \in \boldsymbol{R}, \tag{1}$$

where $0 < \alpha \leqslant 2$ and $\beta > 0$. Note that $C(t)$ is positive definite for the above ranges of α and β, and it is completely monotone for $0 < \alpha \leqslant 1$, $\beta > 0$ [4, 5]. When $\alpha = 2$, $\beta = 1$, one gets the usual Cauchy process.

Recall that a process $X(t)$ is said to be self-similar with self-similarity index κ if

$$X(rt) \triangleq r^{\kappa} X(t) \qquad \text{for} \quad r > 0, \tag{2}$$

where $\triangleq$ denotes equality in joint finite distribution. For some applications, (2) can be too restrictive since quite often scaling property holds only locally for small time intervals. It is also known that a stationary Gaussian random process such as $X(t)$ cannot be a self-similar process [13]. However, $X(t)$ satisfies a weaker self-similar property known as local self-similarity. Here we consider three equivalent definitions of local self-similarity. First, a less common definition [15] which says that a Gaussian stationary process is locally self-similar of order κ if its covariance $C(t)$ satisfies for $t \to 0$,

$$C(t) = 1 - \beta|t|^{\kappa}[1 + O(|t|^{\nu})], \qquad \nu > 0. \tag{3}$$

In the case of the GC process, one has $\kappa = \nu = \alpha$. Note that the class of Gaussian processes which satisfy (3) is also known as Adler processes [16, 17], which include the stretched exponent process [6]. A more intuitive alternative definition is the following. A Gaussian process $X(t)$ is said to be locally self-similar of index κ if

$$X(s) - X(rt) \triangleq r^{\kappa}[X(s) - X(t)], \qquad |t - s| \to 0. \tag{4}$$

The two definitions of local self-similarity given in (3) and (4) are equivalent. Since we are dealing with finite-dimensional distribution of a centred Gaussian process, suffice it to verify for t_1 and $t_2 \to 0$,

$$\begin{aligned}\langle (X(s+rt_1) - X(s))(X(s+rt_2) - X(s))\rangle &= \beta|rt_1|^{\alpha} + \beta|rt_2|^{\alpha} - \beta|r(t_1 - t_2)|^{\alpha} \\ &= \beta r^{\alpha}[|t_1|^{\alpha} + |t_2|^{\alpha} - |t_1 - t_2|^{\alpha}] \\ &= \langle \{r^{\alpha/2}(X(s+t_1) - X(s))\}\{r^{\alpha/2}(X(s+t_2) - X(s))\}\rangle. \end{aligned} \tag{5}$$

Instead of (3) and (4), one can also adopt the definition of locally asymptotically self-similar property first introduced for multifractional Brownian motion $B_{H(t)}(t)$, which is a generalization of fBm $B_H(t)$ with the Hurst index H replaced by $H(t)$, with $0 < H(t) < 1$ [18]. For this purpose, we replace $H(t)$ by a positive constant in the original definition. A Gaussian process $X(t)$ is said to be locally asymptotically self-similar with index κ at a point t_0 if

$$\lim_{\varepsilon\to 0} \frac{X(t_0 + \varepsilon u) - X(t_0)}{\varepsilon^{\kappa}} = T_{t_0}(u), \tag{6}$$

where T_{t_0} is called the tangent process of $X(t)$ at the point t_0. Falconer [19] has shown that if the tangent process for a Gaussian process exists and is non-degenerate, then it is a self-similar Gaussian process with stationary increments. By noting that up to a multiplicative constant fBm is the only Gaussian self-similar process with stationary increments [13], the tangent process (6) is a fBm $B_{\kappa}(u)$ with Hurst index κ. This can easily be verified by direct computation, using (3) with $\kappa = \alpha/2$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left\langle \left(\frac{X(t_0+\varepsilon u) - X(t_0)}{\varepsilon^{\alpha/2}}\right)\left(\frac{X(t_0+\varepsilon v) - X(t_0)}{\varepsilon^{\alpha/2}}\right)\right\rangle \\ &\quad = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{\alpha}}\{\varepsilon^{\alpha}(|u|^{\alpha} + |v|^{\alpha} - |u-v|^{\alpha})\} \sim \langle B_{\alpha/2}(u)B_{\alpha/2}(v)\rangle. \end{aligned} \tag{7}$$

The tangent process of the GC process at the point t_0 is the fBm indexed by $\alpha/2$ (up to a multiplicative constant). In geostatistical application [2], there exists another way to link an Adler process such as the GC process with fBm. It can be shown by using the same method as given in [6] that the GC process forms a locally stationary representation of an allowable linear combination of fBm, namely $\sum_{i=1}^{n} \lambda_i B_H(t_i)$ with $\sum_{i=1}^{n} \lambda_i = 0$.

In order to determine the fractal dimension of the graph of $X(t)$, we consider the local property of the process. The fractal dimension D of a locally self-similar process of order α is given by [15, 16]

$$D = 2 - \frac{\alpha}{2}. \tag{8}$$

Note that the local irregularities of the sample paths are measured by the parameter α, which can be regarded as the fractal index of the process. Thus, the behaviour of the covariance function at the origin to a great extent determines the roughness of the random process.

The GC process is non-Markovian since its covariance $C(t_1, t_2)$ does not satisfy the triangular relation

$$C(t_1, t_3) = C(t_1, t_2)C(t_2, t_3)/C(t_2, t_2), \qquad t_1 < t_2 < t_3, \tag{9}$$

which is a necessary condition for a Gaussian process to be Markovian [20]. In fact, up to a multiplicative constant, the Ornstein–Uhlenbeck process is the only stationary Gaussian

Markov process. Since $X(t)$ has memory, it would be interesting to see whether $X(t)$ is a short- or a long-memory process. For this purpose, we use the following definition [6, 21, 22]. A stationary centred Gaussian process with covariance $C(t)$ is said to be a long-memory process or LRD if

$$\int_0^\infty C(t)\,\mathrm{d}t = \infty. \tag{10}$$

It can be shown that the GC process $X(t)$ is a LRD process if $0 < \alpha\beta \leqslant 1$. One has [23]

$$\int_0^\infty C(t)\,\mathrm{d}t = \int_0^\infty (1+t^\alpha)^{-\beta}\,\mathrm{d}t = \begin{cases} \dfrac{1}{\alpha}B\left(\dfrac{1}{\alpha}, \beta - \dfrac{1}{\alpha}\right), & \text{if } \beta > 0,\ \alpha\beta > 1, \\ \infty, & \text{if } \beta > 0,\ \alpha\beta < 1, \end{cases} \tag{11}$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function and $\Gamma(z)$ is the gamma function. For the limiting case when $\alpha\beta = 1$, one substitutes $\beta = 1/\alpha$ into (11) which gives

$$\int_0^\infty C(t)\,\mathrm{d}t = \frac{\Gamma(1/\alpha)\Gamma(0)}{\alpha\Gamma(1/\alpha)} = \frac{\Gamma(0)}{\alpha}, \tag{12}$$

which diverges since $\Gamma(z) \sim 1/z$ as $z \to 0$. Thus, the GC process is LRD for $0 < \alpha\beta \leqslant 1$ and is short-range dependent (SRD) if $\alpha\beta > 1$.

The large time lag behaviour of the covariance (1) is given by the hyperbolically decaying covariance $C(t) \sim |t|^{-\alpha\beta}, t \to \infty$, which is characteristic of LRD. If the covariance is re-expressed as $(1 + |t|^\alpha)^{-\zeta/\alpha}$, then the parameters α and ζ, respectively, provide separate characterization of fractal dimension and LRD. The separate characterization of the fractal dimension (local property) and LRD (global property) appears to be more natural and flexible than that based on a single parameter such as in fGn. The comparison of the GC process and fGn in the modelling of internet traffic has been studied [25].

3. Lamperti transformation of the generalized Cauchy process

The GC process $X(t)$ is not a self-similar process. However, it is possible to obtain a self-similar process $Y(t)$ from a stationary process $X(t)$ by using the Lamperti transformation [24]. For a stationary process $X(t)$, $t \in \boldsymbol{R}$, and if $H > 0$, we let

$$Y(t) = t^H X(\ln t), \tag{13}$$

for $t > 0$, $Y(0) = 0$, then $Y(t)$ is an H-self-similar (H-ss) process. Conversely, if $\{Y(t),\ t \geqslant 0\}$ is an H-ss, and if we let

$$X(t) = \mathrm{e}^{-Ht}Y(\mathrm{e}^t), \qquad t \in \boldsymbol{R}, \tag{14}$$

then $\{X(t),\ t \in \boldsymbol{R}\}$ is a stationary process. By applying the Lamperti transformation to the GC process $X(t)$ results in a Gaussian H-ss non-stationary process $Y(t)$ with zero mean and covariance:

$$\langle Y(t)Y(s)\rangle = (ts)^H[1 + |\ln(t/s)|^\alpha]^{-\beta}, \qquad t, s > 0. \tag{15}$$

It can be shown by direct computation that the increment process of $Y(t)$ is non-stationary. Now we want to see whether the increment process of $Y(t)$ satisfies a weaker stationary property. It is possible to show that the increments of $Y(t)$ are locally stationary if an additional condition is imposed on the parameter α. We have for $t \gg \tau > 0$,

$$\begin{aligned} \langle (Y(t+\tau) - Y(t))^2\rangle &= \langle (Y(t+\tau))^2\rangle + \langle (Y(t))^2\rangle - 2\langle Y(t+\tau)Y(t)\rangle \\ &= (t+\tau)^{2H} + t^{2H} - 2[t(t+\tau)]^H\left[1 + \left[\ln\left(\frac{t+\tau}{t}\right)\right]^\alpha\right]^{-\beta} \\ &= 2\beta\left(\frac{\tau}{t}\right)^\alpha t^{2H} + O(\max(t^{2H-2}\tau^2, t^{2H-2\alpha}\tau^{2\alpha}, t^{2H-\alpha-1}\tau^{\alpha+1})), \end{aligned} \tag{16}$$

where we have used series expansion. If we let $\alpha = 2H$, then (16) becomes

$$\langle (Y(t+\tau) - Y(t))^2 \rangle \sim \tau^{2H}, \qquad \tau \to 0^+, \tag{17}$$

which implies that the increments of $Y(t)$ are asymptotically locally stationary. Recall that the tangent process of the GC process is fBm indexed by $\alpha/2$ or H if $\alpha = 2H$. Thus, the condition for local stationarity of the Lamperti-transformed GC process is consistent with the local property of the GC process.

Next, we consider the LRD of $Y(t)$. Recall that for a non-stationary process with a correlation function

$$R(t,\ t+\tau) = \frac{C(t,\ t+\tau)}{\sqrt{C(t+\tau,\ t+\tau)C(t,\ t)}}, \tag{18}$$

the condition for LRD is given by [5, 21]

$$\int_0^\infty R(t,\ t+\tau)\,\mathrm{d}\tau = \infty. \tag{19}$$

The correlation of $Y(t)$ is

$$R(t,\ t+\tau) = \left\{1 + \left[\ln\left(1 + \frac{\tau}{t}\right)\right]^\alpha\right\}^{-\beta}, \qquad t, \tau > 0. \tag{20}$$

By noting that for $x > 0$, $\ln(1+x) < x$, we get for $\alpha > 0$, $\beta > 0$,

$$\left\{1 + \left[\ln\left(1 + \frac{\tau}{t}\right)\right]^\alpha\right\}^{-\beta} > \left[1 + \left(\frac{\tau}{t}\right)^\alpha\right]^{-\beta}. \tag{21}$$

Thus,

$$\begin{aligned}\int_0^\infty R(t, t+\tau)\,\mathrm{d}\tau &= \int_0^\infty \left\{1 + \left[\ln\left(1 + \frac{\tau}{t}\right)\right]^\alpha\right\}^{-\beta} \mathrm{d}\tau > \int_0^\infty \left[1 + \left(\frac{\tau}{t}\right)^\alpha\right]^{-\beta} \mathrm{d}\tau \\ &= t\int_0^\infty (1 + z^\alpha)^{-\beta}\,\mathrm{d}z = \infty, \qquad \text{if} \quad \alpha\beta < 1,\end{aligned} \tag{22}$$

where we have used the result from (11). This implies that $Y(t)$ is a LRD process if $\alpha\beta < 1$ and it is a SRD process if $\alpha\beta > 1$. We note that for a large time lag, (20) approaches a form similar to the covariance of one of the LRD stationary processes obtained by Ma [26] by randomizing the time scale of a certain SRD stationary process. We remark that $Y(t)$ provides an example that the application of the Lamperti transformation to a LRD process (in this case the GC process) gives a process which is also LRD. Examples of the Lamperti transformation encountered so far usually relate either two SRD or short-memory processes (for example, the Lamperti transformation between the two Markov processes, namely the Ornstein–Uhlenbeck process and Brownian motion), or a LRD process and a SRD process (in the case of fBm and its Lamperti-transformed process). Thus, the LRD property of the GC process is preserved under the Lamperti transformation. In addition, the conditions of LRD for both the GC process and its Lamperti counterpart are the same, namely $\alpha\beta < 1$. On the other hand, the H-ss process associated with the GC process with $\alpha\beta > 1$ also preserves the SRD property.

4. Asymptotic properties of spectral density

Just like in the case of the stationary Gaussian process with stretched exponential covariance [6], the analytic simplicity of the covariance function of X is not inherited by the corresponding spectral density. Although a closed-form expression for the spectral density of the GC process does not seem to exist, expressions for its asymptotic behaviour can be obtained. These asymptotic expressions are very useful in physical and engineering applications. Kotz *et al* [27] have obtained detailed results for the Linnik probability distribution for the case with $0 < \alpha < 2, \beta = 1$. Erdogan and Ostrovskii [28] have considered the general Linnik distribution with $0 < \alpha < 2, \beta > 1$, including the asymmetric case. These authors used the method of contour integral and series representations of the Linnik distribution to obtain results of its analytic and asymptotic properties. For application purposes, we give a heuristic method which stresses more on practical accessibility rather than mathematical rigour. The main idea is to obtain the asymptotic expansion of the spectral density of the GC process by formally applying term-by-term Fourier transform to the binomial expansion of its covariance function. Such a method allows us to derive many of the important results on the asymptotic expansion of the spectral density of the GC process which were obtained in [28] in terms of the generalized Linnik probability distributions. We would like to stress that the treatment given here is meant for practitioners and it is by no means rigorous. For more complete and rigorous results, one needs to refer to [28].

First, we consider the integral operator with the kernel $|t-u|^{-\nu}$ called the Riesz potential operator [28], which is given by

$$I^{\alpha}\phi(t) = \frac{1}{\Lambda(\alpha)}\int_{-\infty}^{\infty}\frac{\phi(u)\,\mathrm{d}u}{|t-u|^{1-\alpha}}, \qquad \alpha \neq 1+2l, \quad l = 0, 1, 2, \ldots, \tag{23}$$

with

$$\Lambda(\alpha) = \frac{2^{\alpha}\sqrt{\pi}\,\Gamma(\alpha/2)}{\Gamma[(1-\alpha)/2]}. \tag{24}$$

Here we make use of the results that the Riesz kernel of order α in $\boldsymbol{R}$ is given by the Fourier transform of $|t|^{-\alpha}$ [29, 30]:

$$F[|t|^{-\alpha}] = \{[\Lambda(\alpha)]^{-1}|\omega|^{\alpha-1}, \qquad \alpha \neq 1+2l, \quad \alpha \neq -2l, \tag{25a}$$

$$\{[\Lambda_1(\alpha)]^{-1}|\omega|^{\alpha-1}[\Omega_l(\alpha) - \ln|\omega|], \qquad \alpha = 1+2l, \tag{25b}$$

where $F[\cdot]$ denotes the Fourier transform, $l = 0, 1, 2, \ldots,$ and

$$\Lambda_1(\alpha) = (-1)^{(1-\alpha)/2}\sqrt{\pi}\,2^{\alpha-1}\Gamma(\alpha/2)\Gamma[(\alpha+1)/2], \tag{26}$$

$$\Omega_l(\alpha) = \ln 2 + \frac{1}{2}\left[-\gamma + \Gamma'(\alpha/2)/\Gamma(\alpha/2) + \sum_{j=1}^{l}\frac{1}{j}\right], \tag{27}$$

and γ is Euler's constant [23] with value $\gamma = -\Gamma'(1) \approx 0.577\,216$. We remark that the Fourier transform of $|t|^{-\alpha}$ is to be treated in the sense of generalized functions over Schwartz space of test functions ([31], appendix B).

Recall that the function $g(x) = (1+x)^{\nu}$, $\nu \in \boldsymbol{R}$ and $|x| < 1$, can be expanded as a binomial series [32]

$$(1+x)^{\nu} = \sum_{k=0}^{\infty}\binom{\nu}{k}x^k = \sum_{k=0}^{\infty}\frac{\Gamma(\nu+k)}{\Gamma(\nu)\Gamma(1+k)}x^k. \tag{28}$$

If we replace x by $|\tau|^{\alpha}$ with $\alpha > 0$ in (28), then for $|\tau| < 1$ the binomial expansion of the covariance function $(1+|\tau|^{\alpha})^{-\beta}$ is given by

$$C(t) = (1+|t|^{\alpha})^{-\beta} = \sum_{k=0}^{\infty}\binom{-\beta}{k}|t|^{\alpha k}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\beta+k)}{\Gamma(\beta)\Gamma(1+k)}|t|^{\alpha k}. \tag{29}$$

If we formally carry out the term-by-term Fourier transform for the series (29) with the help of (25), we get

$$S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} C(t)\, dt = 2\pi \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\beta + k)}{\Gamma(\beta)\Gamma(1 + k)} \frac{2^{\alpha k}\Gamma[(1 + \alpha k)/2]}{\sqrt{\pi}\Gamma(-\alpha k/2)} |\omega|^{-\alpha k - 1}. \tag{30}$$

Using the following identities for gamma functions ([23], appendix A):

$$\Gamma(2z) = 2^{2z-1}(\pi)^{-1/2}\Gamma(z)\Gamma(z + 1/2) \tag{31}$$

and

$$-z\Gamma(z)\Gamma(-z) = \frac{\pi}{\sin(z\pi)}, \tag{32}$$

(30) can be expressed as

$$S(\omega) = 2\pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Gamma(\beta + k)\Gamma(1 + \alpha k)\sin(\alpha k\pi/2)}{\pi\Gamma(\beta)\Gamma(1 + k)} |\omega|^{-\alpha k - 1}, \qquad \omega \gg 1, \tag{33}$$

which holds for $0 < \alpha \leqslant 2$. This result is in complete agreement with that obtained by Erdogan and Ostrovskii [28]. The spectral density of the GC process has the following asymptotic value:

$$S(\omega) \sim \frac{\beta\,\Gamma(1 + \alpha)\sin(\alpha\pi/2)}{\pi} |\omega|^{-(\alpha+1)} \qquad \text{as} \quad \omega \to \infty. \tag{34}$$

It can be shown [15] that if a process with spectral density satisfies the above asymptotic behaviour, then its covariance satisfies (3) with local self-similar property. However, the converse is not true. One notes that (34) is obtained by considering the leading term of the infinite series (33), which needs to be truncated for numerical studies.

For the spectral density near the origin, its behaviour is more complicated. Kotz *et al* [27] first noted that the behaviour of the spectral density (for $\alpha \in (0, 2]$ and $\beta = 1$) as $\omega \to 0^+$ depends on the arithmetic nature of α. Erdogan and Ostrovskii [28] later showed that this property remains valid in the general case and the arithmetic character of α and β needs to be taken into account when considering $S(\omega)$ near the origin. Our method only allows us to obtain results for the following two cases, namely $\alpha\beta \neq 1$ and $\alpha\beta = 1$, which are quite sufficient for most practical purposes. A detailed discussion on the dependence of the spectral density on the arithmetic nature of α and β is given in [28].

In order to obtain the long-time (or low-frequency) asymptotic expression, we consider the binomial expansion of the covariance for $t > 1$,

$$\begin{aligned} C(t) &= (1 + |t|^{\alpha})^{-\beta} = |t|^{-\alpha\beta}(1 + |t|^{-\alpha})^{-\beta} \\ &= |t|^{-\alpha\beta} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\beta + k)}{\Gamma(\beta)\Gamma(1 + k)} |t|^{-\alpha k} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\beta + k)}{\Gamma(\beta)\Gamma(1 + k)} |t|^{-\alpha(\beta+k)}. \end{aligned} \tag{35}$$

For $0 < \alpha\beta < 1$, the Fourier transform of (35) is

$$S(\omega) = 2\pi \sum_{k=0}^{\infty} (-1)^k \frac{2^{-\alpha(\beta+k)}\Gamma(\beta + k)\Gamma\left(\frac{1-\alpha(\beta+k)}{2}\right)}{\Gamma(\beta)\Gamma(1 + k)\Gamma[\alpha(\beta + k)/2]} |\omega|^{\alpha(\beta+k)-1}. \tag{36}$$

Again, using the identities (31) and (32), we can obtain the following expression for the spectral density:

$$S(\omega) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\beta + k)\sin(\alpha(\beta + k)\pi/2)}{\Gamma(\beta)\Gamma(1 + k)\sin(\alpha(\beta + k)\pi)\Gamma[\alpha(\beta + k)]} |\omega|^{\alpha(\beta+k)-1}. \tag{37}$$

Therefore, in the limit $\omega \to 0^+$, we have the following asymptotic value:

$$S(\omega) \sim \frac{1}{\cos(\alpha\beta\pi/2)\Gamma(\alpha\beta)}|\omega|^{\alpha\beta-1}. \tag{38}$$

For $0 < \alpha\beta < 1$, $S(\omega)$is divergent at the origin, which is the basic feature of the LRD process.

In the case of $\alpha\beta > 1$, the spectral density can be expanded as

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}(1+|t|^\alpha)^{-\beta}\,\mathrm{d}t = 2\int_0^\infty \mathrm{d}t\,(1+t^\alpha)^{-\beta}\cos(\omega t) \\ &= 2\sum_{k=0}^{\infty}\frac{\omega^{2k}}{(2k)!}\int_0^\infty \mathrm{d}t\,(1+t^\alpha)^{-\beta}t^{2k} \\ &= \frac{2}{\alpha\Gamma(\beta)}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma[(1+2k)/\alpha]\,\Gamma[\beta-(1+2k)/\alpha]\omega^{2k}}{(2k)!} \\ &= \frac{2\pi}{\alpha\Gamma(\beta)}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma[(2k+1)/\alpha]\,\omega^{2k}}{\Gamma(2k+1)\sin[(\beta-(2k+1)/\alpha)\pi]\Gamma[1-\beta+(2k+1)/\alpha]}, \end{aligned} \tag{39}$$

where the following integral identity (#3.251 No 11, p 343 in [23]):

$$\int_0^\infty x^{\mu-1}(1+x^\rho)^{-\nu}\,\mathrm{d}x = \frac{1}{\rho}B\left(\frac{\mu}{\rho},\nu-\frac{\mu}{\rho}\right), \qquad \rho>0, \quad \mathrm{Re}(\mu)<0<\rho\,\mathrm{Re}(\nu), \tag{40}$$

and (32) have been used to obtain the above result. Thus, one gets for $\alpha\beta > 1$,

$$S(\omega) \sim \frac{2\pi}{\alpha\Gamma(\beta)}\frac{\Gamma(1/\alpha)}{\sin((\beta-1/\alpha)\pi)\,\Gamma(1-\beta+1/\alpha)} \qquad \text{as} \qquad \omega \to 0^+, \tag{41}$$

which is finite. This is consistent with the short-range-dependent property of the GC process for $\alpha\beta > 1$. We remark that (36) and (39) are obtained by Erdogan and Ostrovskii [28] as residues of contour integrals. However, our heuristic method does not allow us to derive this general result in a rigorous and unifying way. In order to obtain (39), our main guidance is that for $\alpha\beta > 1$ the integral $\int_0^\infty t^{2k}(1+t^\alpha)^{-\beta}\,\mathrm{d}t$ is finite (which is not the case for $\alpha\beta > 1$). Thus, the expansion in cosine term allows us to get the required result for the asymptotic expression for low frequency. We remark that similar expansions are also employed in deriving series for the stable distribution from its characteristic function [33–36].

Note that in the case $\alpha(\beta+k) = 1+2l$, $l = 1, 2, \ldots$, the spectral density can be obtained by using (25*b*):

$$S(\omega) = \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\beta+k)}{\Gamma(\beta)\Gamma(1+k)}\frac{|\omega|^{\alpha(\beta+k)-1}}{\Lambda_1(\alpha(\beta+k))}[\Omega_l(\alpha(\beta+k)) - \ln|\omega|], \tag{42}$$

where Λ_1 and Ω_l are given by (26) and (27), respectively. If $\alpha\beta = 1$, one obtains by using the identity $\Gamma'(1/2)/\Gamma(1/2) = -(\gamma + 2\ln 2)$ the following leading $k = 0$ terms:

$$S(\omega) \sim \frac{1}{\pi}\ln\frac{1}{|\omega|} - \frac{\gamma}{\pi} \qquad \text{as} \quad \omega \to 0^+, \tag{43}$$

which is divergent at the origin.

Finally, we consider an example of spectral density $S(\omega)$ which has a closed analytic form. In the case when $\alpha = 2$ and $\beta > 0$,

$$C(t) = (1+|t|^2)^{-\beta}, \tag{44}$$

then the spectral density of $X(t)$ is given by

$$S(\omega) = \frac{2^{1/2-\beta}}{\sqrt{\pi}\Gamma(\beta)}|\omega|^{\beta-1/2}K_{\beta-1/2}(|\omega|), \tag{45}$$

where $K_\nu(z)$ is the modified Bessel function of second kind (or the MacDonald function) [23, 37]. Here we remark that (45) coincides with the symmetric Bessel distribution, which has (44) as its characteristic function [38]. Since

$$K_\nu(z) \sim 2^{\nu-1}\Gamma(\nu)z^{-\nu} \qquad \text{for} \quad z \to 0^+, \quad \nu > 0, \tag{46}$$

one gets

$$S(\omega) \sim \frac{\Gamma\left(\frac{1}{2}-\beta\right)}{2^{2\beta}\sqrt{\pi}\,\Gamma(\beta)}|\omega|^{2\beta-1} \qquad \text{as} \quad |\omega| \to 0^+. \tag{47}$$

Note that $S(\omega) \to \infty$ if $2\beta < 1$, which is the condition of LRD. For the borderline case of LRD with $2\beta = 1$, by using $K_0(z) \sim \ln(1/z)$ as $z \to 0^+$ one gets the same asymptotic behaviour as (43). The GC process becomes short-range dependent when $2\beta > 1$, with

$$S(\omega) \sim \frac{\Gamma\left(\frac{1}{2}-\beta\right)}{2\sqrt{\pi}\,\Gamma(\beta)} \qquad \text{as} \quad |\omega| \to 0^+. \tag{48}$$

On the other hand, for $z \to \infty$,

$$K_\nu(z) = \sqrt{\frac{\pi}{2}}z^{-1/2}\,\mathrm{e}^{-z}\left(1 - \frac{4\nu^2}{8z} + \cdots\right), \tag{49}$$

which gives

$$S(\omega) \sim \mathrm{e}^{-|\omega|}|\omega|^{\beta-1} \qquad \text{as } |\omega| \to \infty. \tag{50}$$

5. Application to the dielectric relaxation process

In this section, we consider the possible application of the GC process to relaxation phenomena. We show that the covariance function of the GC process can be regarded as a relaxation function in the time domain. To be specific, we consider linear dielectric relaxation, though our discussion applies to other relaxation processes as well.

Experimental evidences indicate that there exists a universal pattern for relaxation behaviour independent of the nature of materials [11]. Such a universal character is reflected in the fractional power laws for a dielectric response function for large and small times:

$$f(t) = -\frac{\mathrm{d}\phi}{\mathrm{d}t} = \begin{cases} (t/\tau)^{-n}, & \text{for} \quad t \ll \tau, \\ (t/\tau)^{-m-1}, & \text{for} \quad t \gg \tau, \end{cases} \tag{51}$$

where we have followed the notations used in the literature of relaxation theory with $\phi(t)$as the relaxation function, $\tau > 0$ denotes the characteristic relaxation time and $0 < m, n < 1$ [11]. In the frequency domain, the complex dielectric susceptibility is given by the one-sided Fourier transform:

$$\chi(\omega) = \chi'(\omega) - \mathrm{i}\chi''(\omega) = \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}(-\phi(t)), \tag{52}$$

which exhibits fractional power behaviour with

$$\begin{aligned} &\chi'(\omega) \propto \chi''(\omega) \propto (\tau\omega)^{n-1}, && \tau\omega \gg 1, \\ &\Delta\chi'(\omega) = \chi'(0) - \chi'(\omega) \propto \chi''(\omega) \propto (\tau\omega)^m, && \tau\omega \ll 1. \end{aligned} \tag{53}$$

One of the most widely used empirical law which reproduces the above asymptotic power law behaviour is the Havriliak and Negami function [11, 12] given by

$$\chi_{\mathrm{HN}}(\omega) = [1 + (\mathrm{i}\tau_{\mathrm{HN}}\omega)^a]^{-b}, \qquad 0 < a, b < 1, \quad \tau_{\mathrm{HN}} > 0, \quad \omega > 0, \tag{54}$$

which satisfies (53) if $a = m$ and $b = (1-n)/m$. For $b = 1$, (54) is known as the Cole–Cole (CC) function, and when $a = 1$ it is the Cole–Davidson (CD) function.

Here we propose a new model for the relaxation function by using the GC process. Physical conditions imposed on the relaxation function require $\phi(t)$ to satisfy the causal condition $\phi(0) = 0$ for $t < 0$, $\phi(t) > 0$ for $t > 0$ and $\phi(t) \to 0$ as $t \to \infty$. It is also convenient to assume that the relaxation function is normalized to $\phi(0) = 1$. The dielectric relaxation function can then be modelled by the one-sided GC process $X(t), t \geqslant 0$ with covariance given by

$$C_+(t) = \theta(t)[1 + (t/\tau_{\mathrm{GC}})^\alpha]^{-\beta} \equiv \theta(t)\phi_{\mathrm{GC}}(t), \tag{55}$$

where the Heaviside step function $\theta(t) = 1$ for $t \geqslant 0$ and is zero otherwise, and $\tau_{\mathrm{GC}} > 0$ is the characteristic relaxation time. The response function based on (55) has the following asymptotic power law behaviour:

$$-\frac{\mathrm{d}\phi_{\mathrm{GC}}}{\mathrm{d}t} \propto \begin{cases} t^{\alpha-1}, & t \ll \tau_{\mathrm{GC}}, \\ t^{-\alpha\beta-1}, & t \gg \tau_{\mathrm{GC}}, \end{cases} \tag{56}$$

which is in agreement with (51) if we let $n = 1 - \alpha$ and $m = \alpha\beta$. The complex susceptibility is then given by the one-sided Fourier transform [11, 12]:

$$\chi(\omega) = -\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\theta(t)\,\mathrm{d}\phi_{\mathrm{GC}}(t) = \frac{\beta}{\tau_{\mathrm{GC}}}\int_0^\infty \left(\frac{t}{\tau_{\mathrm{GC}}}\right)^{\alpha-1}\left[1 - \left(\frac{t}{\tau_{\mathrm{GC}}}\right)^\alpha\right]^{-\beta} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{57}$$

By using the same method as given in the previous section, we expand the integrand as a binomial series for $t < \tau_{\mathrm{GC}}$ and $t > \tau_{\mathrm{GC}}$, and perform term-by-term the one-sided Fourier transform by using the Fourier transform of $t_+^\lambda = \theta(t)t^\lambda$ given by

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} t_+^\lambda\,\mathrm{d}t = \mathrm{i}\Gamma(\lambda+1)\left[\mathrm{e}^{\mathrm{i}\lambda\pi/2}\omega_+^{-\lambda-1} - \mathrm{e}^{-\mathrm{i}\lambda\pi/2}\omega_-^{-\lambda-1}\right], \tag{58}$$

for non-integral λ [31]. For $\omega > 0$, we obtain the following asymptotic expansions of the susceptibility for high and low frequencies, respectively, for non-integers αk and $\alpha(\beta + k)$:

$$\begin{aligned}\chi_{\mathrm{GC}}(\omega) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Gamma(\beta+k)\Gamma(1+\alpha k)}{\Gamma(\beta)\Gamma(k)} \mathrm{e}^{-\mathrm{i}\alpha k\pi/2}(\tau_{\mathrm{GC}}\omega)^{-\alpha k} \\ &\sim \beta\Gamma(1+\alpha)[\cos(\alpha\pi/2) - \mathrm{i}\sin(\alpha\pi/2)](\tau_{\mathrm{GC}}\omega)^{-\alpha}, \qquad \tau_{\mathrm{GC}}\omega \gg 1,\end{aligned} \tag{59}$$

and

$$\begin{aligned}\chi_{\mathrm{GC}}(\omega) &= \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(\beta+k+1)\Gamma(1-\alpha(\beta+k))}{\Gamma(\beta)\Gamma(1+k)} \mathrm{e}^{\mathrm{i}\alpha(\beta+k)\pi/2}(\tau_{\mathrm{GC}}\omega)^{\alpha(\beta+k)} \\ &\sim \beta\Gamma(1-\alpha\beta)\{\cos(\alpha\beta\pi/2) + \mathrm{i}\sin(\alpha\beta\pi/2)](\tau_{\mathrm{GC}}\omega)^{\alpha\beta}, \qquad \tau_{\mathrm{GC}}\omega \ll 1.\end{aligned} \tag{60}$$

In the case with $\alpha = 1$, and for $t < \tau_{\mathrm{GC}}$, instead of (58) one uses

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} t_+^k\,\mathrm{d}t = (-\mathrm{i})^k\pi\delta^{(k)}(\omega) + k!PV(\mathrm{i}\omega)^{-(k+1)}, \qquad k = 1, 2, \ldots, \tag{61}$$

where PV(·) denotes principal value. Alternatively, one can employ the integral identity

$$\int_0^\infty \mathrm{e}^{zx}(1+x)^{-\nu}\,\mathrm{d}x = z^{-1+\nu/2}\,\mathrm{e}^{z/2}W_{-\nu/2,(1-\nu)/2}(z), \tag{62}$$

where $W_{\lambda,\mu}(z)$ is Whittaker's function [23]. Using the asymptotic expansions of $W_{\lambda,\mu}(z)$ for $z \to +\infty$ and $z \to 0^+$, one obtains

$$\chi_{\mathrm{GC}}(\omega) \sim \beta(\mathrm{i}\tau_{\mathrm{GC}}\omega)^{-1}, \qquad \tau_{\mathrm{GC}}\omega \gg 1, \tag{63}$$

$$\chi_{\mathrm{GC}}(\omega) \sim \beta\Gamma(-\beta)(\mathrm{i}\tau_{\mathrm{GC}}\omega)^{\beta}, \qquad \tau_{\mathrm{GC}}\omega \ll 1, \tag{64}$$

which agree, respectively, with (59) and (60) with $\alpha = 1$.

The above discussion shows that the real and imaginary parts of the complex susceptibility $\chi_{\rm GC}(\omega)$ scale asymptotically like

$$\begin{aligned} &\chi'_{\rm GC}(\omega) \propto \chi''_{\rm GC}(\omega) \propto \omega^{-\alpha}, \qquad \tau_{\rm GC}\omega \gg 1, \\ &\Delta\chi'_{\rm GC}(\omega) = \chi'_{\rm GC}(0) - \chi'_{\rm GC}(\omega) \propto \chi''_{\rm GC}(\omega) \propto \omega^{\alpha\beta}, \qquad \tau_{\rm GC}\omega \ll 1, \end{aligned} \tag{65}$$

with their ratio given by

$$\chi''_{\rm GC}(\omega)/\chi'_{\rm GC}(\omega) = \tan(\alpha\pi/2), \qquad \tau_{\rm GC}\omega \gg 1, \tag{66}$$

$$\chi''_{\rm GC}(\omega)/\Delta\chi'_{\rm GC}(\omega) = \tan(\alpha\beta\pi/2), \qquad \tau_{\rm GC}\omega \ll 1. \tag{67}$$

This asymptotic behaviour is similar but not the same as that of the HN function (54). By letting $m = a$ and $(1-n)/m = b$ in (53) and comparing it with (59) and (60) give the conditions for the asymptotic power laws of the HN model to be the same as that of the GC model as $\alpha = ab$ and $\beta = 1/b$, which lead to contradiction with the assumptions that a, b, α and β only take values in (0, 1]. In the case of the CC function with $b = 1$, then both $\phi_{\rm CC}(t)$ and $\phi_{\rm GC}(t)$ have the same asymptotic laws if we take $\beta = 1$.

The comparison between the GC model and the HN model can also be carried out in the time domain. Bertelsen [39] was the first to derive the response function that corresponds to the HN function, which allows one to obtain the long-time asymptotic power law. Recently, Hilfer [40] has obtained the relaxation function associated with the HN function in terms of the Fox function. For our comparison purpose, we can derive the relaxation function $\phi_{\rm HN}(t)$ by taking the one-sided inverse Fourier transform of (54). We remark that due care needs to be taken for the CD case in order to obtain the correct asymptotic expression. In the high-frequency region, the term-by-term one-sided inverse Fourier transform of the series expansion of (53) gives for non-integer $a(b+k)$,

$$\int_0^\infty \chi_{\rm HN}(\omega)\,{\rm e}^{{\rm i}\omega t}\,{\rm d}\omega = -\frac{{\rm d}\phi_{\rm HN}}{{\rm d}t} = \sum_{k=0}^\infty (-1)^k \frac{\Gamma(b+k)(t/\tau_{\rm HN})^{a(b+k)-1}}{\tau_{\rm HN}\Gamma(1+k)\Gamma(b)\Gamma[a(b+k)]} \tag{68}$$

or

$$\phi_{\rm HN}(t) = 1 + \sum_{k=0}^\infty (-1)^{k+1} \frac{\Gamma(b+k)(t/\tau_{\rm HN})^{a(b+k)}}{\Gamma(1+k)\Gamma(b)\Gamma[1+a(b+k)]}, \tag{69}$$

where we have used $\phi_{\rm HN}(0) = 1$ and $z\Gamma(z) = \Gamma(1+z)$. From (68), one gets for $t \ll \tau_{\rm HN}$, $\frac{{\rm d}\phi_{\rm HN}}{{\rm d}t} \sim t^{ab-1}$. In the case of the CC function with $b = 1$, we get from (68)

$$\frac{{\rm d}\phi_{\rm CC}}{{\rm d}t} = \sum_{k=0}^\infty (-1)^{k+1} \frac{(t/\tau_{\rm CC})^{a(1+k)-1}}{\tau_{\rm CC}\Gamma[a(1+k)]} = \frac{-1}{\tau_{\rm CC}}\left(\frac{t^{a-1}}{\tau_{\rm CC}}\right) E_{a,a}(-(t/\tau_{\rm CC})^a), \tag{70}$$

where $E_{a,b}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(ak+b)}$ is the generalized Mittag–Leffler function [41], when $b = 1, E_{a,1}(z) \equiv E_a(z)$ is the Mittag–Leffler function. Thus, subject to the condition $\phi_{\rm CC}(0) = 1$,

$$\phi_{\rm CC}(t) = 1 + \sum_{k=1}^\infty (-1)^k \frac{(t/\tau_{\rm CC})^{ak}}{\Gamma(1+ak)} = E_a(-(t/\tau_{\rm CC})^a). \tag{71}$$

By using the small-time asymptotic value of the Mittag–Leffler function $E_a(z) \sim 1 - \frac{z}{\Gamma(1+a)}$ (or the leading term of the series of (71)), one gets $\frac{{\rm d}\phi_{\rm CC}(t)}{{\rm d}t} \sim t^{a-1}$ for $t \ll \tau_{\rm CC}$. When $a = 1$, the CD function $\chi_{\rm CD}(\omega) = [1 + {\rm i}\tau_{\rm CD}\omega]^{-b}$ is the one-sided Fourier transform of

$$-\frac{{\rm d}\phi_{\rm CD}(t)}{{\rm d}t} = \frac{(t/\tau_{\rm CD})^{b-1}\,{\rm e}^{-t/\tau_{\rm CD}}}{\tau_{\rm CD}\Gamma(b)}, \tag{72}$$

which gives for $t \ll \tau_{CD}$, $\frac{d\phi_{CD}}{dt} \sim t^{-(1-b)}$. (72) can also be obtained from (69) by letting $a = 1$. Imposing condition $\phi_{CD}(0) = 1$, the CD relaxation function for short time is given by

$$\phi_{CD}(t) = 1 - \frac{1}{\Gamma(b)} \int_0^t (s/\tau_{CD})^{b-1} \, e^{-(s/\tau_{CD})} \, d(s/\tau_{CD}) = 1 - \frac{\gamma(b, t/\tau_{CD})}{\Gamma(b)}, \tag{73}$$

where $\gamma(b, z)$ is the incomplete gamma function [23]. Note that (72) coincides with the gamma distribution, with the CD function corresponding to its characteristic function [38]. The series expansion of (73) is given by (68) with $a = 1$.

Similarly, one can derive the long-time asymptote of the HN response function by taking the one-sided inverse Fourier transform of the series expansion of (54) in the low-frequency region for non-integer ak:

$$\phi_{HN}(t) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma(b+k)(t/\tau_{HN})^{-ak}}{\Gamma(1+k)\Gamma(b)\Gamma(1-ak)}, \tag{74}$$

such that $\phi_{HN}(t) \sim t^{-a}$ for $t \gg \tau_{HN}$. For $b = 1$, one gets the CC relaxation function $\phi_{CC}(t) \sim t^{a}$ for $t \gg \tau_{CC}$, which again can be obtained by using the large-time asymptotic property of the Mittag–Leffler function with $E_a(z) \sim \frac{z^{-1}}{\Gamma(1-a)}$ in (71). Note that in the case of the CD relaxation function with $a = 1$, if we consider the asymptotic expansion of $\gamma(b, t/\tau_{CD})$, we obtain for $t \gg \tau_{CD}$,

$$\phi_{CD}(t) = \frac{(t/\tau_{CD})^{b-1} \, e^{-t/\tau_{CD}}}{\Gamma(b)} \sum_{k=0}^{\infty} \prod_{l=1}^{k} \frac{b-l}{(t/\tau_{CD})^{l}}. \tag{75}$$

However, in order to get the correct asymptotic expression for $\phi_{CD}(t)$ (or $-d\phi_{CD}(t)/dt$) corresponding to $\chi'_{CD}(0) - \chi'_{CD}(\omega) \propto \chi''_{CD}(\omega) \propto \tau_{CD}\omega, \tau\omega \ll 1$, one considers instead of (61) the following Fourier pair [31]:

$$F\left[t_+^{-k}\right] = \frac{(i\omega)^{k-1}}{(k-1)!} \left[\sum_{n=1}^{k-1} n^{-1} + \Gamma'(1) + i\pi/2 - \ln(\omega + i0) \right]. \tag{76}$$

(76) implies that $d\phi_{CD}/dt \sim t^{-2}$ and $\phi_{CD} \sim t^{-1}$ for $t \gg \tau_{CD}$.

Again, by comparing the results obtained for various cases of $\phi_{HN}(t)$ with the corresponding values for $\phi_{GC}(t)$, we conclude that only in the case of the Cole–Cole function with $b = \beta = 1$ they have the same asymptotic fractional power laws for both the large- and short-time regions. The Cole–Davidson function $\phi_{CD}(t)$ deviates considerably from $\phi_{GC}(t)$ in the large-time region. We have tabulated the results in table 1.

Next we want to compare ϕ_{GC} with another popular relaxation model based on the Kohlrausch–Williams–Watts (KWW) function or the stretched exponential relaxation function (in time domain) [11, 12] given by

$$\phi_{KWW}(t) = \exp[-(t/\tau_{KWW})^{\mu}], \qquad 0 < \mu < 1, \quad t \geqslant 0, \quad \tau_{KWW} > 0. \tag{77}$$

(77) has the same functional form as the one-sided positive stable (or Levy) distribution. For high frequencies, both $\chi_{GC}(\omega)$ and $\chi_{KWW}(\omega)$ have same fractional power dependence if $\mu = \alpha$. This is to be expected since for sufficiently small time, both $\phi_{GC}(t)$ and $\phi_{KWW}(t)$ have similar local behaviour with appropriate choice of parameters (figure 1). Both these functions satisfy locally self-similar property [6]. However, $\phi_{GC}(t)$ differs from $\phi_{KWW}(t)$ considerably for large t (figure 2). In fact, it can be shown in the low-frequency limit that $\Delta\chi'_{KWW}(\omega) \sim \omega^2$ and $\chi''_{KWW}(\omega) \sim \omega$, which are the same as $\Delta\chi'_{DC}(\omega)$ and $\chi''_{DC}(\omega)$, but different from the results (62) and (63). Hence, in the frequency domain the KWW relaxation function differs from the HN (except for the Cole–Davidson case) and the GC model in the low-frequency limit

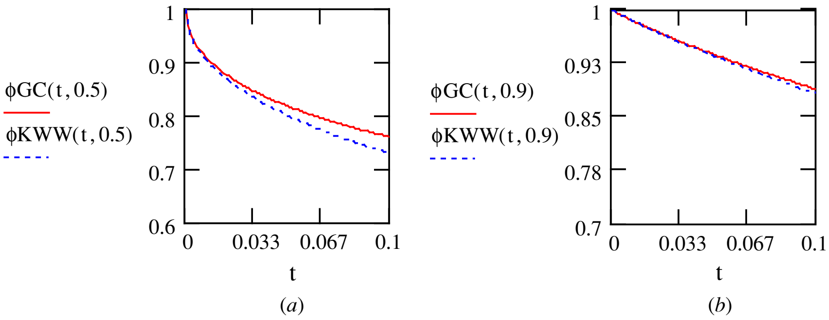

Figure 1. Comparing $\phi_{\mathrm{GC}}(t)$ and $\phi_{\mathrm{KWW}}(t)$ with $\tau_{\mathrm{GC}} = \tau_{\mathrm{KWW}} = 1$ for $t \ll 1$. (*a*) $\alpha = \mu = 0.5$; (*b*) $\alpha = \mu = 0.9$.

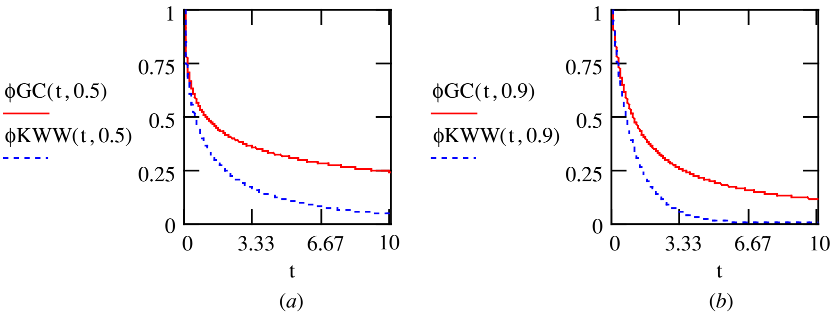

Figure 2. Comparing $\phi_{\mathrm{GC}}(t)$ and $\phi_{\mathrm{KWW}}(t)$ with $\tau_{\mathrm{GC}} = \tau_{\mathrm{KWW}} = 1$ for $t \gg 1$. (*a*) $\alpha = \mu = 0.5$; (*b*) $\alpha = \mu = 0.9$.

Table 1. Summary of asymptotic properties of response functions and susceptibility functions of GC, HN, CC, CD and KWW models. For each function, the upper row gives long-time and low-frequency asymptotes and the lower row gives short-time and high-frequency asymptotes.

	$-\frac{d\phi}{dt}$	$\chi(\omega)$	$\chi'(\omega)$	$\chi''(\omega)$	$\Delta\chi'(\omega)$
GC	$t^{-\alpha\beta-1}$	$\omega^{\alpha\beta}$	–	$\omega^{\alpha\beta}$	$\omega^{\alpha\beta}$
	$t^{\alpha-1}$	$\omega^{-\alpha}$	$\omega^{-\alpha}$	$\omega^{-\alpha}$	–
HN	t^{-a-1}	ω^{a}	–	ω^{a}	ω^{a}
	t^{ab-1}	ω^{-ab}	ω^{-ab}	ω^{-ab}	–
CC	t^{-a-1}	ω^{a}	–	ω^{a}	ω^{a}
	t^{a-1}	ω^{-a}	ω^{-a}	ω^{-a}	–
CD	t^{-2}	ω	–	ω	ω^{2}
	t^{b-1}	ω^{-b}	ω^{-b}	ω^{-b}	–
KWW	$t^{\mu-1}\,\mathrm{e}^{-t^{\mu}}$	ω	–	ω	ω^{2}
	$t^{\mu-1}$	$\omega^{-\mu}$	$\omega^{-\mu}$	$\omega^{-\mu}$	–

(see table 1). There exists one main difference between the stretched exponential relaxation function and that of GC and HN. The stochastic process defined by the covariance $\phi_{\mathrm{KWW}}(t)$ is a SRD or short-memory process since

$$\int_0^\infty \exp -(t/\tau_{\mathrm{KWW}})^{\mu}\,\mathrm{d}t = \frac{\tau_{\mathrm{KWW}}}{\mu}\Gamma(1/\mu) < \infty. \qquad (78)$$

On the other hand, both $\phi_{GC}(t)$ and $\phi_{HN}(t)$ are covariances of LRD processes for $0 < \alpha\beta \leqslant 1$ and $0 < ab \leqslant 1$.

The above discussion shows that it is possible to use the one-sided GC covariance function $\phi_{GC}(t)$ to model the dielectric relaxation function, and it is expected to provide a reasonable good fit to experimental data that can be represented by $\phi_{HN}(t)$ (in particular, the Cole–Cole relaxation function $\phi_{CC}(t)$) and the KWW function. This preliminary consideration needs to be followed by a comparison of the GC model with the experimental data to actually test the model.

Despite the similarity in the functional forms of the generalized Cauchy (GC) relaxation function and the Havriliak–Negami (HN) function, these two functions differ in several physical aspects. First, we note that one of them (the GC function) is in the time domain while the other (the HN function) is defined in the frequency domain. They are not Fourier transforms of each other, there does not seem to exist an analytic link between these two functions. One clear distinction between these two functions is that for the GC case, the relaxation function in time domain has a closed analytic form, but the susceptibility does not, whereas the HN function has a closed form in frequency domain but not its relaxation function. As mentioned in our earlier discussion, although the asymptotic power laws based on these two functions are similar, they are not exactly the same. In practice, one may attempt to distinguish these two functions at the intermediate time and frequency ranges. Furthermore, it is important to link the two parameters in these functions with physical properties of the relaxation medium. It was pointed out by Coffey [42] recently that one of the most important question in non-Debye (or anomalous) dielectric relaxation is the physical interpretation of the parameters α and β (or a and b) in various relaxation formulae, and what are the physical conditions which give rise to these parameters. In the case of the GC function, if it is regarded as the covariance of a random process, then the two parameters α and β can be given the following physical interpretation. The parameter α determines the fractal dimension and self-similarity (or to be more exact, local self-similarity) of the process, while β characterizes the long-range dependence (or long-memory property). On the other hand, since the inverse Fourier transform of the HN function does not have a close form, we can only obtain the time domain relaxation function in the form of asymptotic series. Its large-time behaviour still indicates long-range dependence, and its fractal character can be seen from its small-time power law behaviour. It is clear that more physical considerations are necessary to trace the origin and the basis of the relaxation functions, which may provide a clue to the understanding of the relationship between them.

6. Conclusion

We have studied some of the basic properties of the two-parameter generalized Cauchy process. The asymptotic properties of its spectral density are obtained based on the Fourier transform of the series expansion of the covariance function. The results obtained are in agreement with that of Kotz *et al* [27] and Erdogan and Ostrovskii [28], who employed the contour integral method. We also consider the possible application of the GC process to relaxation phenomena. It is shown that the covariance of the one-sided generalized Cauchy process gives an asymptotic power law behaviour of the response function of the relaxation process similar to the HN function. It is only in the case of the Cole–Cole function ($b = 1$) does the GC relaxation function (with $\beta = 1$) have the same fractional power laws for both high- and low-frequency domains. In comparison with the KWW function, the GC relaxation function shows similarity only in the short-time or high-frequency region.

We note that there exist some attempts to provide a stochastic formulation to the relaxation process (see [43] for a brief review), in particular, for the CC and KWW models. In view of the fact that $\phi_{GC}(t)$, $\phi_{CC}(t)$ and $\phi_{KWW}(t)$ have several basic common properties such as they are completely monotone and infinitely divisible [4, 5], it will be interesting to see whether a similar stochastic formulation can also be obtained for the GC model. In particular, one may follow the approach of Kalmykov *et al* [44] to obtain a microscopic model for the HN function based on continuous time random walk and fractional Fokker–Planck equation. A stochastic formulation based on the knowledge of physical properties of the materials may provide an explanation to the 'universal' character of these relaxation functions [11].

Acknowledgments

The first author would like to thank the Malaysian Ministry of Science, Technology and Innovation for IRPA grant no. 09-99-01-0095-EA093. The second author would like to acknowledge the support given by the National Natural Science Foundation of China (NSFC) under the project grant number 60573125.

Appendix A. Fourier transforms of powers of *t* [29–31]

Let S be Schwartz space of rapidly decreasing infinitely differentiable test functions and S' be the space of linear continuous functionals on S, which are called tempered distributions (or tempered generalized functions) over S. A tempered distribution f defines a continuous functional in S' via

$$(f, \varphi) = \int f(t)\varphi(t)\,\mathrm{d}t, \qquad \varphi \in S. \tag{A.1}$$

If we denote Fourier transform by $F[\cdot]$, then the Fourier transform $F[f]$ of any tempered distribution f is defined by

$$(F[f], \varphi) = (f, F[\varphi]), \qquad f \in S', \quad \varphi \in S, \tag{A.2}$$

and

$$F[\varphi](\omega) = \int_R \varphi(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t, \qquad \varphi \in S. \tag{A.3}$$

Bearing the above definitions in mind, we give the Fourier transforms of generalized functions t_+^λ and $|t|^\lambda$ below:

$$F\left[t_+^\kappa\right](\omega) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\lambda(\pi/2)}\Gamma(\lambda+1)(\omega+\mathrm{i}0)^{-\lambda-1}, \tag{A.4}$$

for $\lambda \neq -1, -2, \ldots$. If $\lambda \neq 0, \pm 1, \pm 2, \ldots$,

$$F\left[t_+^\lambda\right](\omega) = \mathrm{i}\Gamma(\lambda+1)\left[\mathrm{e}^{\mathrm{i}\lambda\pi/2}\omega_+^{-\lambda-1} - \mathrm{e}^{-\mathrm{i}\lambda\pi/2}\omega_-^{-\lambda-1}\right]. \tag{A.5}$$

For $\lambda = n$, a non-negative integer,

$$F\left[t_+^n\right](\omega) = [(-\mathrm{i})^n\pi\delta^{(n)}(\omega) + n!PV(\mathrm{i}\omega)^{-n-1}], \tag{A.6}$$

where PV$(\cdot)$ denotes the principal value.

Fourier transform of $|t|^{-\lambda}$ is given by

$$F[|t|^{-\lambda}](\omega) = \begin{cases} [\Lambda(\lambda)]^{-1}|\omega|^{\lambda-1}, & \lambda \neq 1+2l, \quad \kappa \neq -2l, \\ [\Lambda_1(\lambda)]^{-1}|\omega|^{\lambda-1}[\Omega_l(\lambda) - \ln|\omega|], & \lambda = 1+2l, \end{cases} \tag{A.7}$$

where $l = 0, 1, 2, \ldots$, and

$$\Lambda_1(\lambda) = (-1)^{(1-\lambda)/2}\sqrt{\pi}2^{\lambda-1}\Gamma(\lambda/2)\Gamma[(\lambda+1)/2], \tag{A.8}$$

$$\Omega_l(\lambda) = \ln 2 + \frac{1}{2}\left[-\gamma + \Gamma'(\lambda/2)/\Gamma(\lambda/2) + \sum_{j=1}^{l}\frac{1}{j}\right], \tag{A.9}$$

and γ is Euler's constant [22] with value $\gamma = -\Gamma'(1) \approx 0.577\,216$.

For t_+^{-n}, we have

$$F\left[t_+^{-n}\right](\omega) = \frac{(\mathrm{i}\omega)^{n-1}}{\Gamma(k)}\left[\sum_{n=1}^{k-1} n^{-1} + \Gamma'(1) + \mathrm{i}\pi/2 - \ln(\omega + \mathrm{i}0)\right]. \tag{A.10}$$

Appendix B. Some functional relations of gamma functions ([23], p 946)

Below are some functional relations of gamma functions which have been used in this paper:

$$\Gamma(1+x) = x\Gamma(x), \tag{B.1}$$

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x}, \tag{B.2}$$

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \frac{\pi}{\cos \pi x}, \tag{B.3}$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(\frac{1}{2}+x\right). \tag{B.4}$$

The following are gamma functions for some particular values:

$$\Gamma(1) = \Gamma(2) = 1, \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \tag{B.5}$$

$$\Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}. \tag{B.6}$$

For n a natural number,

$$\Gamma(1+n) = n!. \tag{B.7}$$

References

[1] Gneiting T and Schlather M 2004 *SIAM Rev.* **46** 269
[2] Chiles J-P and Delfiner P 1999 *Geostatistics, Modeling Spatial Uncertainty* (New York: Wiley)
[3] Linnik V Ju 1963 *Sel. Transl. Math. Stat. Probab.* **3** 1
[4] Pillai R N 1990 *Ann. Inst. Stat. Math.* **42** 157
[5] Lin G D 1998 *J. Stat. Plan. Inference* **74** 1
[6] Lim S C and Muniandy S V 2003 *J. Phys. A: Math. Gen.* **36** 3961
[7] Kozubowski T J, Podgorski K and Samoroditsky G 1998 *Extremes* **1** 367
[8] Huillet T 2000 *J. Phys. A: Math. Gen.* **33** 2631
[9] Kozubowski T J and Rachev S T 2001 *Eur. J. Oper. Res.* **74** 310
[10] Barndorff-Nielsen O E, Mikosch T and Resnick S I (ed) 2001 *Levy Processes, Theory and Applications* (Boston, MA: Birkhauser)
[11] Jonscher A K 1996 *Universal Relaxation Law* (London: Chelsea)
[12] Havriliak S and Havriliak S J 1997 *Dielectric and Mechanical Relaxation in Materials: Analysis, Interpretation, and Application to Polymers* (Munich: Carl Hanser)

[13] Samorodnitsky G and Taqqu M 1994 *Stable Non-Gaussian Random Processes* (New York: Chapman and Hall)
[14] Paxson V and Floyd S 1995 *IEEE/ACM Trans. Netw.* **3** 226
[15] Kent J T and Wood A T A 1997 *J. R. Stat. Soc.* B **59** 579
[16] Adler A J 1981 *The Geometry of Random Fields* (New York: Wiley)
[17] Lang G and Roueff F 2001 *Stat. Inference Stoch. Process.* **4** 283
[18] Benassi A, Jaffard S and Roux D 1997 *Rev. Mat. Iberoamericana* **13** 19
[19] Falconer K J 2003 *J. Lond. Math. Soc.* **67** 657
[20] Todorovic P 1992 *An Introduction to Stochastic Processes and Their Applications* (New York: Springer) p 100
[21] Mantegna R and Stanley H E 2000 *An Introduction to Econophysics, Correlation and Complexity in Finance* (Cambridge: Cambridge University Press)
[22] Rangarajan G and Ding M (ed) 2003 *Processes with Long-Range Correlations, Theory and Applications* (New York: Springer)
[23] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series and Products* (San Diego, CA: Academic)
[24] Li M and Lim S C 2005 *Mod. Phys. Lett.* B **19** 829
[25] Lamperti J W 1962 *Trans. Am. Math. Soc.* **104** 62
[26] Ma J C 2003 *J. Appl. Probab.* **40** 1133
[27] Kotz S, Ostrovskii I V and Hayfavi A 1995 *J. Math. Anal. Appl.* **193** 353, 497
[28] Erdogan M B and Ostrovskii I V 1998 *J. Math. Anal. Appl.* **217** 555
[29] Samko S G, Kilbas A A and Maritchev D I 1993 *Integrals and Derivatives of the Fractional Order and Some of Their Applications* (Amsterdam: Gordon and Breach)
[30] Samko S G 2002 *Hypersingualr Integrals and Their Applications* (London: Taylor and Francis)
[31] Gelfand I M and Vilenkin K 1964 *Generalized Functions* vol 1 (New York: Academic)
[32] Wade W R 2004 *An Introduction to Analysis* 3rd edn (Upper Saddle River, NJ: Prentice-Hall) p 215
[33] Montroll E W and Bender J T 1984 *J. Stat. Phys.* **34** 129
[34] Bender J T 1984 *J. Stat. Phys.* **36** 625
[35] Zolotarev V M 1986 *One-Dimensional Stable Distributions* (Providence, RI: American Mathematical Society) p 94
[36] Uchaikin V V and Zolotarev V M 1999 *Chance and Stability, Stable Distributions and Their Applications* (Utrecht: VSP BV) p 103, chapter 4
[37] Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic) p 250
[38] Kotz S, Kozubowski T J and Podgorski K 2001 *The Laplace Distribution and Generalizations* (Boston, MA: Birkhauser)
[39] Bertelsen J V 1970 *Mat.-Fys. Medd. K. Dan. Vidensk. Selsk.* **38** 1
[40] Hilfer R 2002 *Phys. Rev.* E **65** 061510
[41] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1955 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
[42] Coffey W T 2004 *J. Mol. Liq.* **114** 5
[43] Jonscher A K, Jurlewicz A and Weron K 2003 *Contemp. Phys.* **44** 329
[44] Kalmykov Y P, Coffey W T, Crothers D S F and Titov S V 2004 *Phys. Rev.* E **70** 041103